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THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

By F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

[Continued from September-October Number.]

IV. Typical of the Cassinian Oval, we have by the *Method of Complex Variables* the following equations :

$$x = \beta \sqrt{1 + \alpha t}, = \beta \sqrt{1 + \alpha(\cos \theta + i \sin \theta)} \dots\dots (A);$$

$$y = \beta \sqrt{1 + \alpha / t}, = \beta \sqrt{1 + \alpha(\cos \theta - i \sin \theta)} \dots\dots (B).$$

$$\therefore dx/dt = \frac{1}{2} i^2 \alpha \beta (\cos \theta - i \sin \theta) / \sqrt{1 + \alpha(\cos \theta + i \sin \theta)} \dots\dots (A');$$

$$dy/dt = \frac{1}{2} i^2 \alpha \beta (\cos \theta + i \sin \theta) / \sqrt{1 + \alpha(\cos \theta - i \sin \theta)} \dots\dots (B').$$

$$\therefore \left(\frac{d\mathbf{P}_1}{d\theta} \right)^2 = \frac{i^4 \alpha^2 \beta^2 (\cos \theta - i \sin \theta)(\cos \theta + i \sin \theta)}{4 \sqrt{\{ [1 + \alpha(\cos \theta + i \sin \theta)] \times [1 + \alpha(\cos \theta - i \sin \theta)] \}} \dots\dots (C).$$

$$\therefore \mathbf{P} = \alpha \beta \int_0^{2\pi} \frac{d\theta}{[1 + \alpha^2 + 2\alpha \cos \theta]^{\frac{1}{2}}} = \frac{2\alpha \beta}{[1 + \alpha^2]^{\frac{1}{2}}} \int_0^\pi \frac{d\theta}{[1 + M \cos \theta]^{\frac{1}{2}}} \dots\dots (D).$$

[From (A) and (B), after differentiating with respect to t , we have

$$dx/dt = \frac{1}{2} \alpha \beta / x = \frac{1}{2} \alpha \beta / \sqrt{1 + \alpha t},$$

$$dy/dt = -\frac{1}{2} \alpha \beta / t^2 y = -\frac{1}{2} \alpha \beta / t^2 \sqrt{1 + \alpha / t}.$$

$$\therefore (d\mathbf{P}_1)^2 = \left(\frac{-\alpha^2 \beta^2}{4 \sqrt{[(1 + \alpha t)(1 + \alpha / t)]}} \right) \left(\frac{dt}{t} \right)^2 \dots\dots (E).$$

Differentiating under the assumption that $t = \cos \theta + i \sin \theta$, etc.,

$$\frac{dt}{t} = \left(\frac{i \cos \theta - \sin \theta}{\cos \theta + i \sin \theta} \right) d\theta = i \left(\frac{\cos \theta + i \sin \theta}{\cos \theta + i \sin \theta} \right) d\theta = i d\theta.$$

$$\therefore \mathbf{P}_1 = \frac{1}{2} i \alpha \beta \int \frac{dt}{t [1 + \alpha^2 + \alpha(t + 1/t)]^{\frac{1}{2}}} = \frac{1}{2} \alpha \beta \int \frac{d\theta}{[1 + \alpha^2 + 2\alpha \cos \theta]^{\frac{1}{2}}};$$

$$\text{and } \mathbf{P} = \frac{1}{2} \alpha \beta \int_0^{4\pi} \frac{d\theta}{[1 + \alpha^2 + 2\alpha \cos \theta]^{\frac{1}{2}}} = \frac{2\alpha\beta}{[1 + \alpha^2]^{\frac{1}{2}}} \int_0^{\pi} \frac{d\theta}{[1 + M \cos \theta]^{\frac{1}{2}}}. \quad]$$

Since in the Cassinian Oval under consideration, $\alpha = \frac{5}{4}$ and $\beta = 2$, we have $M = \frac{4}{3}$; that is, (D) is *identical* with (4) on page 265 of the September-October MONTHLY. Slowly converging series may be obtained by transforming under the hypothesis that $\theta = (90 - \phi)$, or under the hypothesis that $\theta = (90 + \phi)$.

V. The assumption of (2) from page 264 of the MONTHLY specified gives

$$\mathbf{P} = m^2 \int_b^a \frac{r^2 dr}{\sqrt{\{[(m^2 + c^2)^2 - r^4] \times [r^4 - (m^2 - c^2)^2]\}}} \dots\dots\dots (2).$$

Let $(m^2 - c^2) / (m^2 + c^2) = e^2$, and $r^2 = (m^2 + c^2)x^2 \dots\dots\dots (k)$;

$$\text{then } \mathbf{P} = \frac{4m^2}{\sqrt{(m^2 + c^2)}} \int_e^1 \frac{2(m^2 + c^2)^2 x^2 dx}{\sqrt{\{[(m^2 + c^2)^2(1 - x^4)] \times [(m^2 + c^2)^2(x^4 - e^4)]\}}} \dots\dots\dots (\delta),$$

$$\text{or } \mathbf{P} = \frac{4m^2}{\sqrt{(m^2 + c^2)}} \int_e^1 \frac{2x^2 dx}{\sqrt{[(1 - x^4)(x^4 - e^4)]}} \dots\dots\dots (F),$$

$$= \frac{4m^2}{\sqrt{(m^2 + c^2)}} \left[\int_e^1 \frac{(x^2 - e) dx}{\sqrt{[(1 - x^4)(x^4 - e^4)]}} + \int_e^1 \frac{(x^2 + e) dx}{\sqrt{[(1 - x^4)(x^4 - e^4)]}} \right] \dots\dots\dots (G),$$

$$= \sqrt{\left\{ \left(\frac{16m^4}{m^2 + c^2} \right) \left(\frac{1}{2(1 + e^2)} \right) \right\}} \left[cn^{-1} \left(\frac{x + e/x}{1 + e}, \frac{1 + e}{\sqrt{2(1 + e^2)}} \right) \right.$$

$$\left. + cn^{-1} \left(\frac{x - e/x}{1 - e}, \frac{1 - e}{\sqrt{2(1 + e^2)}} \right) \right]_e^1 = \sqrt{\left\{ \left(\frac{16m^4}{m^2 + c^2} \right) \left(\frac{1}{2(1 + e^2)} \right) \right\}} \left[\{ cn^{-1}(+1) \right.$$

$$\left. - cn^{-1}(+1) \} + \{ cn^{-1}(-1) - cn^{-1}(+1) \} \right] \dots\dots\dots (H);$$

that is, the *first* indicated integral in (G) has *vanished*. The expression for the perimeter of the *Cassinian Oval*, therefore, becomes

$$\mathbf{P} = 2\pi m \left\{ 1 + \sum \left(\frac{1.3.5.7 \dots (2n-1)}{2.4.6.8 \dots 2n} \right)^2 \left[\sqrt{\left(1 + \frac{c^2}{m^2} \right)} - \sqrt{\left(1 - \frac{c^2}{m^2} \right)} \right]^{2n} \right\},$$

which is a *complete* elliptic integral of the *first* order.

For the perimeter of the *Bernoullian Lemniscate*, we have $m = c$; that is, symmetrically expressed,

$$\mathbf{P}' = 2\pi c \left\{ 1 + \sum \left(\frac{1.3.5.7 \dots (2n-1)}{2.4.6.8 \dots 2n} \right)^2 \left[\sqrt{\left(1 + \frac{c^2}{c^2} \right)} - \sqrt{\left(1 - \frac{c^2}{c^2} \right)} \right]^{2n} \right\}.$$

For the perimeter of the two *Ovaliform Figures*, we have $m < c$; that is, similarly expressed,

$$P'' \sqrt{2\pi c} \left(\frac{m}{c}\right)^2 \left\{ 1 + \sum \left(\frac{1.3.5.7 \dots (2n-1)}{2.4.6.8 \dots 2n} \right)^2 \left[\sqrt{\left(1 + \frac{m^2}{c^2}\right)} - \sqrt{\left(1 - \frac{m^2}{c^2}\right)} \right]^{2n} \right\}.$$

[Concluded.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by D. P. WAGONER, A. B., Principal of the School of Language, Westerville, Ohio.

A man bought a farm for \$6000 and agreed to pay for it in four equal annual installments, at 6 per cent. annual interest compounded every instant. Required the annual payment.

B. F. Burleson.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana, College, Texarkana, Arkansas-Texas; P. S. BERG, Larimore, North Dakota; and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let $a = \$6000$, $r = .06$, $x =$ annual payment, and $m = 4 =$ number of equal annual payments. If the interest is compounded n times a year, we have the present value of the first installment $= x(1 + \frac{r}{n})^{-n} = xe^{-r}$ when n is infinite; of the second, $= xe^{-2r}$; of the third, $= xe^{-3r}$; of the m th, $= xe^{-mr}$; where $e =$ Napierian base.

(See Todhunter's Differential Calculus, page 136).

$$\therefore a = x \left(\frac{1}{e^r} + \frac{1}{e^{2r}} + \frac{1}{e^{3r}} + \dots + \frac{1}{e^{mr}} \right) = \frac{x}{e^{mr}} \left(\frac{e^{mr} - 1}{e^r - 1} \right)$$

$$\therefore x = \frac{a e^{mr} (e^r - 1)}{e^{mr} - 1} = \frac{a(e^r - 1)}{1 - e^{-mr}} = \frac{a(e^r - 1)}{1 - e^{-4r}}$$

$$\therefore x = \$6000 \left(\frac{e^{.06} - 1}{1 - e^{-.24}} \right) = \$1738.269.$$

II. Solution by B. F. BURLESON, Oneida Castle, New York.

The amount of P in n years at $r = 6\%$ when the interest is compounded q times a year is evidently